For $\alpha \neq 0$, a Raman process is in competition with the Orbach exponential process. As before, in Figs. 4 and 5, the vertical marks give the lower limits of pure exponential laws.

In the case of rare-earth ethylsulphates ($\Theta_D \sim 60°K$), for measurements between 1.5 and 4°K, the interesting region of our curves is in the $15 < \Theta_D/T < 40$ range. For a paramagnetic ion in MgO ($\Theta_D \sim 820°$), the transitional temperature between the two processes is, for $\alpha = 0.5$, about 40°K.

3. CONCLUSION

The study of double-quantum spin-lattice relaxation in a particular three-level system, leads to a relaxation law of the form

$$T_1^{-1} \sim K I_n\left(\frac{\Theta_D}{T}\right) = K e^{-\alpha\Theta_D/T}\left(\frac{T}{\Theta_D}\right)^n \times \int_0^{\Theta_D/T} \frac{e^x x^{n-4}(x-\alpha\Theta_D/T)^3}{(e^x-1)(e^{x-\alpha\Theta_D/T}-1)} dx\,,$$

where K is a constant, $n=7$ for non-Kramers salts, or $n=9$ for Kramers salts, and where the parameter $\alpha = \Theta_c/\Theta_D$ describes the contact between the studied three-level system and the phonon spectrum characterized by its Debye temperature.

Calculations show that the ordinary Raman and Orbach processes are derived as particular cases.

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Stopping Power of Matter for Deuterons at Extreme Relativistic Energies*

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The stopping power of matter for deuterons at extreme relativistic energies ($\lesssim$2000 GeV) has been calculated. The structure of this particle and its spin are taken into account explicitly. It is found that the ultrarelativistic effects reduce the stopping power (as predicted by the relativistic formula) by 8% at the highest energies considered. These effects are analyzed numerically and compared with the estimated density correction. A stopping-power table for deuterons in aluminum is computed.

THE stopping power of matter for protons and muons (spin $\frac{1}{2}$) at extreme relativistic energies was calculated[1] by taking into account the particles's spin, anomalous magnetic moment, and distributions of charge and magnetic moment (particle form factors). In this paper we extend this work to include the deuteron (spin 1).

The differential cross section for scattering of an electron (charge $-e$, rest mass m, and velocity v) at an angle θ from a spinless point particle (charge ze and rest mass M_d), initially at rest, is[2]

$$\left(\frac{d\sigma}{d\Omega}\right)_0 = \left(\frac{ze^2}{2\gamma m v^2}\right)^2 \frac{\cos^2(\frac{1}{2}\theta)}{\sin^4(\frac{1}{2}\theta)}\left(1+\frac{2\gamma m}{M_d}\sin^2\tfrac{1}{2}\theta\right)^{-1}, \quad (1)$$

where $\gamma = (1-\beta^2)^{-1/2}$, $\beta = v/c$ being the speed of the electron in terms of the speed of light c. Gourdin[3] derived an expression for the scattering of electrons from deuterons by using a nonrelativistic wave function for the deuteron. We write[4,5]

$$\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega}\right)_0 \left[\frac{G_{Ed}^2(q^2)}{1+\tau} + \frac{8\tau^2}{9}\frac{G_{Qd}^2(q^2)}{1+\tau} + \frac{2\tau}{3} G_{Md}^2(q^2)\left(\frac{1}{1+\tau} + 2\tan^2\tfrac{1}{2}\theta\right)\right], \quad (2)$$

where $\hbar q$ is the magnitude of the change in the electron's (=deuteron's) energy-momentum four-vector; $\tau = \hbar^2 q^2/4M_d^2c^2$, M_d being the mass of the deuteron; and

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G_{Ed}, G_{Qd}, and G_{Md} are the form factors associated with the distributions of electric dipole moment, electric quadrupole moment, and magnetic dipole moment. The factor G_{Ed} can be written[4]

$$G_{Ed}(q^2)=2G_{ES}(q^2)\int_0^\infty [u^2(r)+w^2(r)]j_0(\tfrac{1}{2}qr)\,dr. \quad (3)$$

Here

$$G_{ES}=\tfrac{1}{2}(G_{Ep}+G_{En}), \quad (4)$$

where G_{Ep} and G_{En} are the electric form factors of the proton and neutron, u and w are the S- and D-state radial functions of the deuteron ground state, and j_n denotes the spherical Bessel function of order n. The quantity G_{Qd} in Eq. (2) is given by

$$G_{Qd}(q^2)=\frac{12M_d c^2}{\hbar^2 q^2}G_{ES}(q^2)\times\int_0^\infty [\sqrt{2}u(r)w(r)-\tfrac{1}{2}w^2(r)]j_2(\tfrac{1}{2}qr)\,dr \quad (5)$$

and

$$G_{Md}(q^2)=\frac{2M_d}{M}\bigg(\int_0^\infty \{G_{MS}[u^2(r)-\tfrac{1}{2}w^2(r)]+\tfrac{3}{4}G_{ES}w^2(r)\}\times j_0(\tfrac{1}{2}qr)\,dr+\int_0^\infty \{(\sqrt{2})^{-1}G_{MS}[u(r)w(r)+(\sqrt{2})^{-1}w^2(r)]+\tfrac{3}{4}G_{ES}w^2(r)\}\,j_2(\tfrac{1}{2}qr)\,dr\bigg), \quad (6)$$

where M is the rest mass of the nucleon and

$$G_{MS}=\tfrac{1}{2}(G_{Mp}+G_{Mn}), \quad (7)$$

G_{Mp} and G_{Mn} being the magnetic form factors of the proton and neutron.

A number of simplifications can be made for the purpose of calculating the extreme relativistic contributions to the stopping-power formula for the deuteron when $q\lesssim 4F^{-1}$. Since the energy lost by the deuteron in a single close collision with an electron is

$$Q=\hbar^2q^2/2m \quad (8)$$

and since the maximum energy that can be lost is

$$Q_m=\frac{2\gamma^2mv^2}{1+2\gamma m/M_d}, \quad (9)$$

this restriction implies that $Q_m\lesssim 610$ GeV and that $\gamma\lesssim 890$. As found below [Eqs. (18) and (19)], however, only a small contribution is made to the stopping power by terms whose accuracy depends on γ being thus restricted. The stopping-power formula obtained below is accurate up to a value of γ of several thousand. With $\gamma\lesssim 10^3$ (deuteron energies $E\lesssim 2000$ GeV), $\tau\lesssim 0.04$ and $1+\tau\cong 1$. Jankus[6] has shown that the term containing the quadrupole form factor in Eq. (2) then reduces to $(8\tau^2/9)(\frac{1}{2}q)^4Q_d^2$, where $Q_d=0.274\ F^2$ is the deuteron's quadrupole moment. When $q=4F^{-1}$ this term has a magnitude 0.002, and so we neglect its presence in Eq. (2). Following Jankus, we also neglect all integrals involving $w(r)$ or $w^2(r)$ in Eqs. (3), (5), and (6), since the ^3D part of the wave function accounts for only about 4–7% of the charge distribution. Furthermore, it has been shown[7,8] that $G_{En}/G_{Ep}\ll 1$ when $q^2\leq 8$, and so we have $G_{ES}=\frac{1}{2}G_{Ep}$. Since, approximately, $G_{Mp}=\mu_p G_{Ep}$ and $G_{Mn}=\mu_n G_{Ep}$,[4] where μ_p and μ_n are the magnetic moments of the proton and neutron expressed in nuclear magnetons, Eqs. (4) and (7) imply that

$$G_{MS}=\tfrac{1}{2}(\mu_p+\mu_n)G_{Ep}=(\mu_p+\mu_n)G_{ES}. \quad (10)$$

Introducing these approximations into Eq. (2), we obtain

$$\frac{d\sigma}{d\Omega}=\left(\frac{d\sigma}{d\Omega}\right)_0\left(1+\frac{2M_d^2\tau}{3M^2}(\mu_p+\mu_n)^2(1+\tan^2\tfrac{1}{2}\theta)\right)\times\left(\int_0^\infty u^2(r)j_0(\tfrac{1}{2}qr)\,dr\right)^2 G_{Ep}^2. \quad (11)$$

Each of these factors is next written in terms of the deuteron's energy loss, Eq. (8). The dependence of $d\sigma$ on the angle θ in Eqs. (1) and (11) can be expressed as a function of Q by the method used in Ref. 1 and $d\sigma$ expressed in terms of the differential dQ. To evaluate the integral in (11) we use the empirical fit,[9]

$$u(r)=9.20(e^{-0.232r}-e^{-1.202r}), \quad (12)$$

with r measured in F and $\int_0^\infty u^2\,dr=1$, and $j_0(x)=(\sin x)/x$ to obtain

$$V\equiv\int_0^\infty u^2(r)j_0(\tfrac{1}{2}qr)\,dr=\frac{1.692}{q}\left(\tan^{-1}\frac{q}{0.928}+\tan^{-1}\frac{q}{4.808}-2\tan^{-1}\frac{q}{2.868}\right), \quad (13)$$

where q is in F^{-1}. For G_{Ep} we use the empirical formula suggested by Hand, Miller, and Wilson[10]:

$$G_{Ep}(q^2)=-\frac{1.24}{1+q^2/30}+\frac{1.34}{1+q^2/14.5}+\frac{0.90}{1+q^2/15.8}. \quad (14)$$

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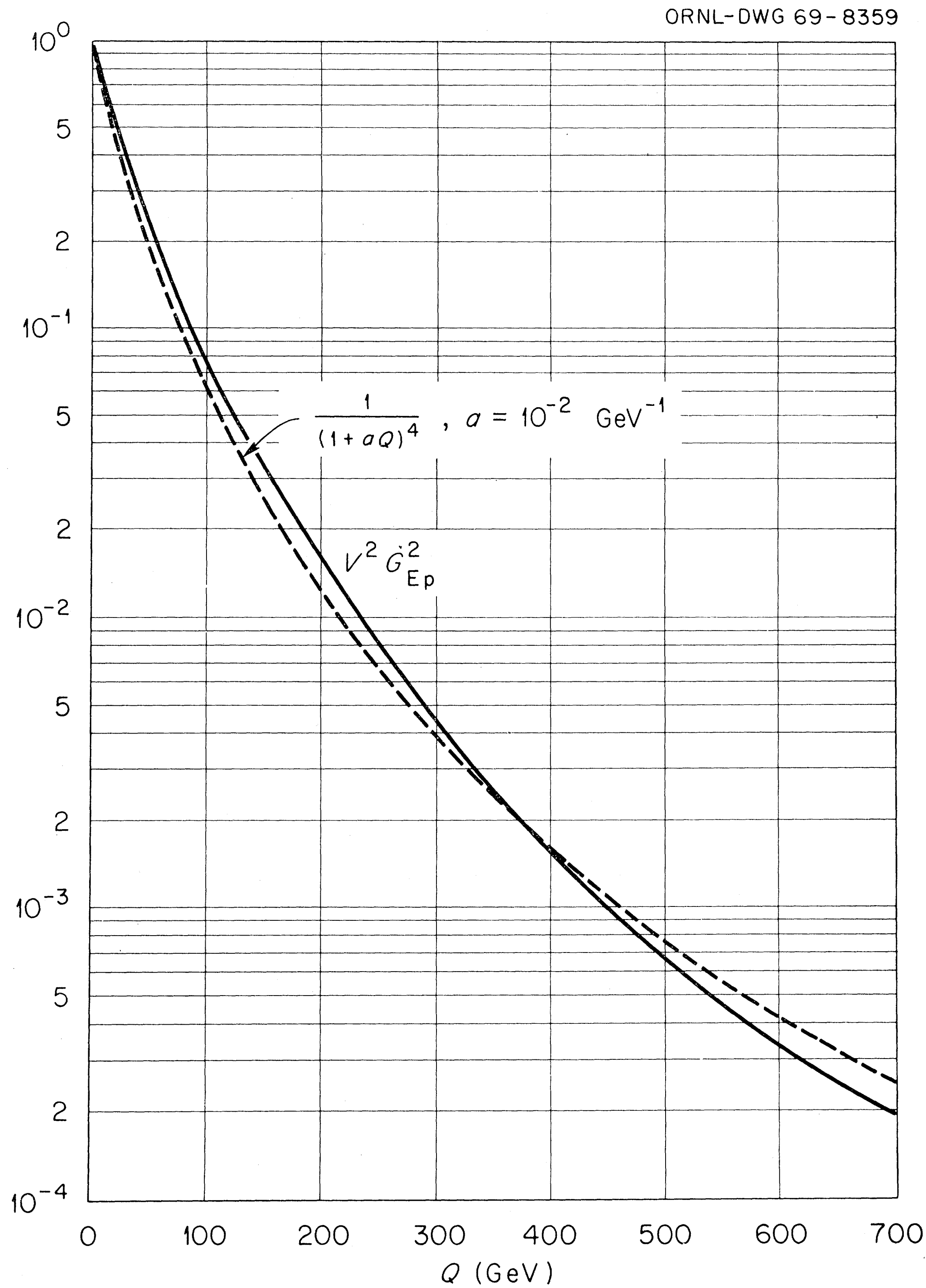

FIG. 1. Comparison of $V^2 G_{Ep}^2$ (solid curve) with empirical fit (dashed curve) given by Eq. (16).

TABLE I. Analysis of contributions to the stopping power of aluminum ($I=163$ eV and density $\rho=2.71$ gm/cm^3) for deuterons at extreme relativistic energies.

γ	Deuteron energy (GeV)	ϵ_0	ϵ_1	ϵ_2 Eq. (19)	ϵ_3 Eq. (19)	ϵ Eq. (18)	δ	$\frac{dE}{\rho ds}$ (MeV cm^2/g)
10	18.8	12.35	12.35	12.34	12.34	12.34	−0.72	1.843
50	93.8	15.57	15.55	15.50	15.50	15.52	−1.9	2.292
100	188	16.95	16.93	16.73	16.73	16.83	−2.5	2.474
250	469	18.79	18.72	···	···	18.32	−3.4	2.708
500	938	20.17	20.06	···	···	19.20	−4.1	2.838
750	1407	20.98	20.82	···	···	19.65	−4.5	2.904
1000	1876	21.56	21.34	···	···	19.95	−4.8	2.948

We now replace Eq. (11) with

$$d\sigma=\frac{2\pi z^2e^4}{mv^2}\frac{dQ}{Q^2}\left[1-\left(\frac{\beta^2}{Q_m}-\frac{m(\mu_p+\mu_n)^2}{3M^2c^2}\right)\times Q-\frac{m(\mu_p+\mu_n)^2}{3M^2c^2}\left(\frac{\beta^2}{Q_m}-\frac{1}{\gamma^2mc^2}\right)Q^2\right]V^2G_{Ep}{}^2. \quad (15)$$

The product $V^2G_{Ep}{}^2$ can be approximated by the function

$$V^2G_{Ep}{}^2=1/(1+aQ)^4, \quad (16)$$

with $a=10^{-2}$ GeV^{-1}, as shown in Fig. 1. Substituting (16) into (15) then gives an expression of the form

$$d\sigma=\frac{2\pi z^2e^4}{mv^2}\left(\frac{1}{Q^2(1+aQ)^4}-\frac{A}{Q(1+aQ)^4}-\frac{B}{(1+Qa)^4}\right)dQ, \quad (17)$$

where A and B are coefficients multiplying Q and Q^2 in Eq. (15).

The contribution of distant collisions to the stopping power of a medium is given by

$$(-dE/ds)_{Q<\eta}=\tfrac{1}{2}\kappa[\ln(2\gamma^2mv^2\eta/I^2)-\beta^2],$$

where $\kappa=4\pi z^2e^4NZ/mv^2$, NZ is the number of electrons per unit volume, and I is the mean excitation energy of the medium. The contribution of close collisions is given by $(-dE/ds)_{Q>\eta}=NZ\int_\eta^{Q_m}Q\,d\sigma$. Multiplying both sides of Eq. (17) by NZQ, integrating, and adding the result to $(-dE/ds)_{Q<\eta}$, we find for the stopping power ($a\eta\ll1$)

$$-\frac{dE}{ds}=\kappa\left[\ln\frac{2\gamma^2mv^2}{I}-\tfrac{1}{2}\beta^2+\tfrac{1}{2}\ln\frac{M_d}{M_d+2\gamma m}-\tfrac{1}{2}\ln(1+aQ_m)-\tfrac{1}{6}\left(\tfrac{11}{2}+\frac{A}{a}+\frac{B}{2a^2}\right)+\frac{1}{2(1+aQ_m)}+\frac{1}{4(1+aQ_m)^2}\times\left(1+\frac{B}{a^2}\right)+\frac{1}{6(1+aQ_m)^3}\left(1+\frac{A}{a}-\frac{B}{a^2}\right)\right]. \quad (18)$$

This formula, which is applicable when $\gamma\lesssim1000$, can be greatly simplified if we assume that $aQ_m\ll1$. Restricting ourselves further to $aQ_m\lesssim0.1$, for which $\gamma\lesssim100$ and $E\lesssim200$ GeV, we obtain from Eq. (18)

$$-\frac{dE}{ds}=\kappa\left[\ln\frac{2\gamma^2mv^2}{I}-\beta^2+\tfrac{1}{2}\ln\frac{M_d}{M_d+2\gamma m}-2aQ_m+\frac{m(\mu_p+\mu_n)^2Q_m}{6M^2c^2}\right]. \quad (19)$$

The relative contributions of the various terms are given in Table I, calculated for Al. The dimensionless quantity $\epsilon=(-dE/ds)/\kappa$ is the entire term in square brackets in Eq. (18); ϵ_0 represents the first two terms in the bracket of (19) and is thus proportional to the stopping power calculated by the ordinary relativistic formula. The quantity $\epsilon_1=\epsilon_0-\frac{1}{2}\ln(1+2\gamma m/M_d)$, which represents the first three terms in the bracket of (19) and is correct at all γ, is proportional to the deuteron's stopping power with the appropriate ultrarelativistic expression (9) used for Q_m. This term is kinematic in origin and appears in the stopping-power formula for any particle of mass M when $\gamma m/M$ is not neglected compared with unity. The quantity $\epsilon_2=\epsilon_1-2aQ_m$ takes into account the deuteron's structure (when $\gamma\lesssim100$). Finally, ϵ_3, which is the complete expression in square brackets in (19), takes into account also the deuteron's spin ($\gamma\lesssim100$).[11] The columns labeled with ϵ's in Table I show that the deuteron's structure and spin decrease the stopping power more than the use of the exact expression (9) for Q_m. Comparison of ϵ_2 and ϵ_3 at the lower energies reveals that the effect of spin there is negligible. The ultrarelativistic corrections to the stopping power amount to 2.5% when $\gamma=100$ and to 8% when $\gamma=10^3$. By comparison, the density correction[12] δ is at least several times larger throughout this range. The last column of Table I gives the mass stopping power of Al as calculated from Eq. (18), δ not being included.

[11] Differences in values of the ϵ's in Table I are the same for any element.

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